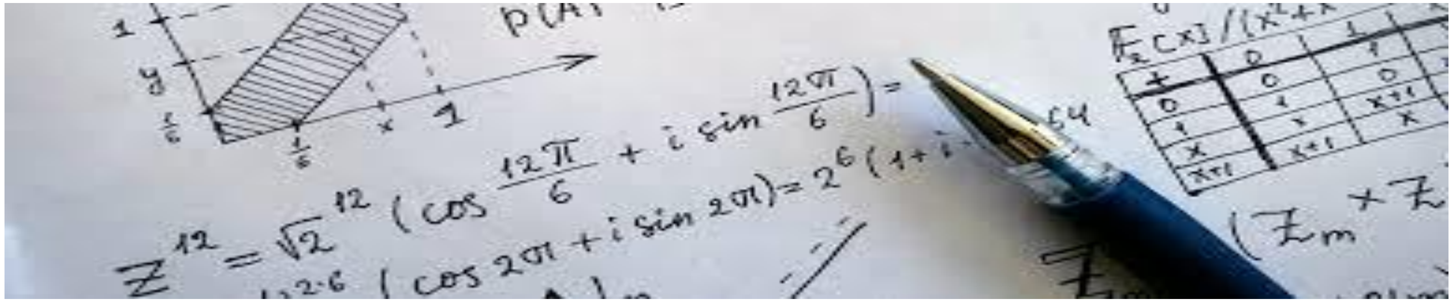

PBL FROM BACHELOR TO ENGINEER

(About how to use one PBL in subjects

Mathematics I at the bachelor degree of study and

Applied Mathematics at the master degree of study)

SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA

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INSTITUTE OF MATHEMATICS AND PHYSICS

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University of La Laguna

Pythagoras
2021-1-RO01-KA220-HED-000032258

PBL – Why we need it in teaching mathematics?

- PBLs demonstrate where and how mathematics can be used continually in special technical subjects
- students learn to work with

	Mathematics	Technical subject
different variables	$x \longrightarrow y(x)$	$t \longrightarrow u(\mathrm{t})$
constants denoted generally	g = 9,81 m/s^2	g (= 9,81 m/s^2)
different notation of differential equation	$y'' = y''(x)$	$\dfrac{d^2u}{dt^2} = u''(t) = \ddot{u}(t)$

PBL – Why we need it in teaching mathematics?

- students become aware of the parallel in the conceptual apparatus of Mathematics and technical subject

Mathematics	Technical subject
the stationary points of a function	the equilibrium position of a system
eigenvalues of the matrix	the stability of a system with forced oscillations
transformation of the motion equation into a system of linear differential equations of first order	transformation of the motion equation into a state space

- students better understand the importance of numerical methods in engineering education
- students have opportunity to test „acquired knowledge“ directly on practical applied problems

⇓

These problems must be introduced into teaching mathematics

since the first year of study at the university

PBL – Where can we find practical problems?

It is necessary to select problems for PBL from the lecture notes for specialised subjects taught at the faculty.

Why?

The advantage of these applications is

that once in the further study

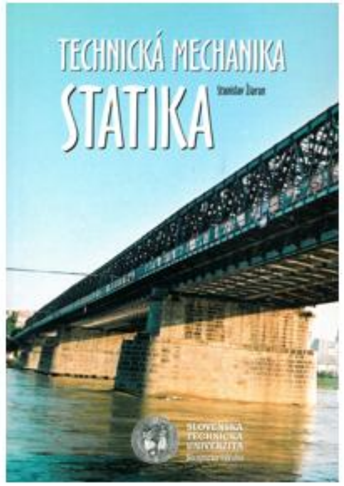
students will “remember them”,

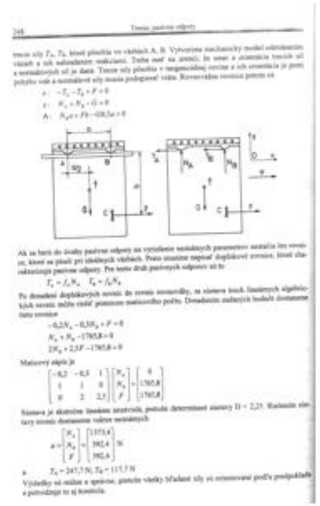
for example – special technical text book

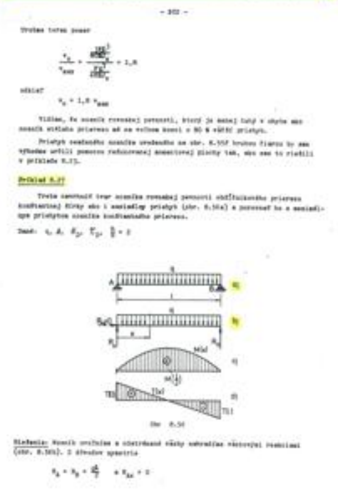
for study programme

Applied Mechanics and Mechatronics

at the FME STU in Bratislava

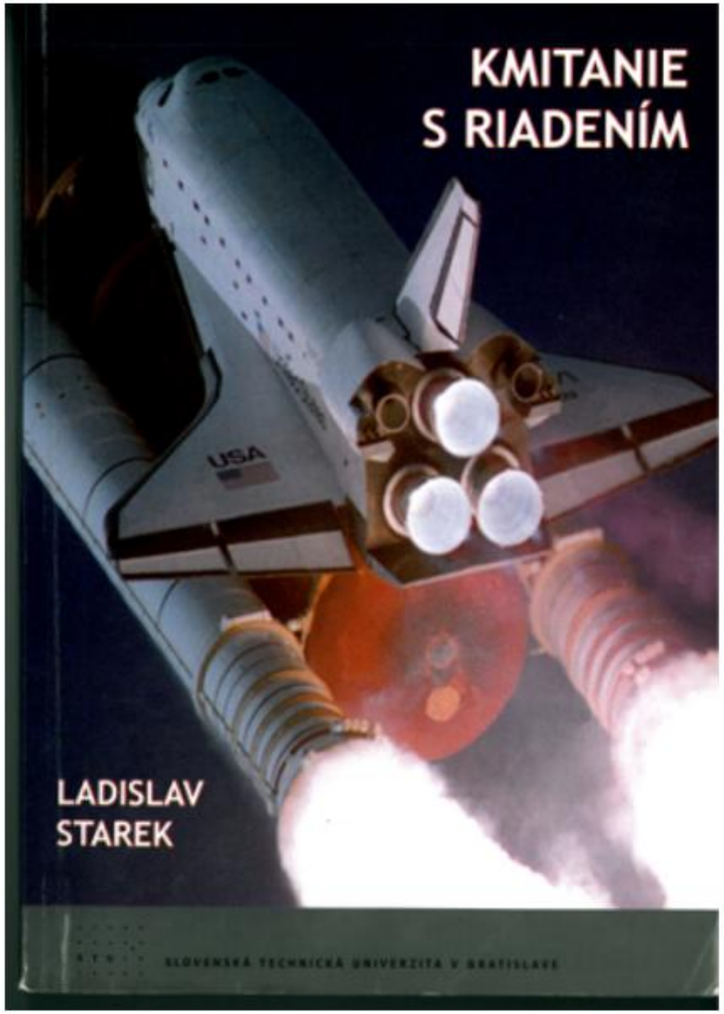






PBL FROM BACHELOR TO ENGINEER

The following PBLs were selected from this lecture notes.



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Ak predpokladáme, že $u = U_a e^{j\omega t}$, bude

$$m\ddot{x} + b\dot{x} + kx = (j\omega b + k)U_a e^{j\omega t} \qquad (3.16)$$

Ak porovnáme túto rovnicu s rovnicou (3.3), vidíme, že obidve rovnice budú rovnaké, ak položíme $F_a = U_a(j\omega b + k)$ a $\varphi = 0$. Preto podľa (3.6a) bude

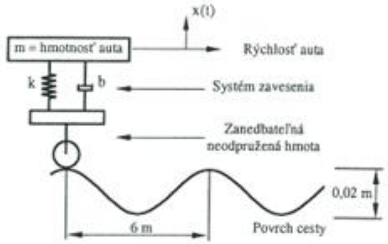
$$\frac{|X|}{U_a} = \sqrt{\frac{1+(2\eta b_p)^2}{(1-\eta^2)^2 + (2\eta b_p)^2}} \qquad (3.17)$$

Vzťah (3.17) vyjadruje pomer maximálnej amplitúdy odozvy k amplitúde vstupu, resp. odpruženej hmoty v závislosti od naladenia.

Príklad 3.2

Bežným príkladom kmitania s kinematickým budením je pohyb automobilu po nerovnej ceste, resp. pohyb lietadla po rozbehovej dráhe. V tomto príklade treba vyšetriť vplyv rýchlosti a hmotnosti športového automobilu na amplitúdu kmitania. Ekvivalentné hodnoty tuhosti a tlmenia odpruženia automobilu sú 4.10^5 N/m, 2.10^4 Ns/m hmotnosť je 1007 kg. Matematický model cesty je daný vzťahom $y(t) = 0{,}01 \sin \omega_b t$. Frekvencia budenia od cesty je $f_b = v/\lambda$, kde v je rýchlosť automobilu v km/h a λ je vlnová dĺžka nerovnosti v km. Platí, že $f = \omega/2\pi$, a po dosadení máme

$$\omega_b = 2\pi v/\lambda = 2\pi v/(3600.0{,}006) = 0{,}291 \text{ rad/s}$$



Obr. 3.9 Štvrtinový mechanický model automobilu pohybujúci sa konštantnou rýchlosťou po nerovnej ceste (nerovnosti cesty sú pre jednoduchosť aproximované sinusoidou)

Pre rýchlosť automobilu $v = 20$ km/h frekvencia budenia od cesty má hodnotu $\omega_b = 5{,}818$. VUF automobilu je

$$\Omega_0 = \sqrt{4.10^5/1007} = 19{,}93 \text{ rad/s}$$

a teda naladenie

$$\eta = 5{,}818/19{,}93 = 0{,}292$$

Pomerné tlmenie má potom hodnotu

Safe driving of passengers traveling on uneven road

Travellers in cars are exposed to fatigue and unwanted vibrations that affect their health. Driver fatigue affects his performance and reactions, which increases the risk of a traffic accident. Undesirable vibrations, the frequency of which depends on the speed of the car, occur when the car drives on an uneven road (a road with a different profile)

– in the vertical direction (causing floating)

– in the longitudinal direction (causing rocking)

– in the transverse direction (causing swinging).

For vertical direction of oscillation, it is necessary to avoid the **frequency band** in the range of **4-8 Hz** (the natural frequency of the human organism in the abdomen), which causes nauseacking

The safety of the car and the comfort of passengers are improved by the suspension system of the car and properly designed damping.

Several models of the car - quarter, half and full are used in order to simulate the oscillation of the car, it’s non spring-loaded and spring-loaded masses when driving on an uneven road at different speeds.

The behavior of the car when driving on an uneven road allows us to know the solution of the equations of motion for the corresponding model of the car in the physical and state space.

T A S K

A sports car weighing m = 1000 kg moves along an uneven road in time T at a constant speed. Mathematical model of the uneven road is given by function $u(t)$ = sin(t). Equivalent values of the stiffness and damping of the automobile suspension are k = 2000 N/m and b = 2000 Ns/m.

Mathematics I
(**bachelor degree** of study)

Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

$$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$$
$$x(0)=0$$
$$\dot{x}(0)=0$$

where

$x = x(t)$ – a deflection of a car moving on an uneven road

$x' = x'(t)$, $x'' = x''(t)$, $u = u(t)$

Applied Mathematics
(**master degree** of study)

Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

$$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$$
$$x(0)=0$$
$$\dot{x}(0)=30$$

where

$x = x(t)$ – a deflection of a car moving on an uneven road

$x' = x'(t)$, $x'' = x''(t)$, $u = u(t)$

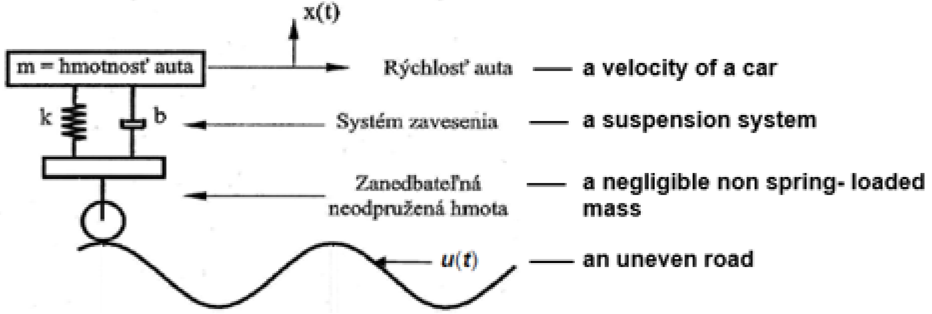


Fig. Quarter mechanical model (with 1 degree of freedom) of a car moving on uneven road

Mathematics I

Task 1.
Adjust the motion equation to the basic form.

Task 2.
Find the deflection of a car moving on an uneven road in time *T*.

Task 3.
Interpret the results about a car moving on an uneven road in physical units

$$x(3) = 0{,}54 \;\; ; \;\; \dot{x}(3) = -1{,}07 \;\; ; \;\; \ddot{x}(3) = -0{,}64$$

Applied Mathematics

The most important tasks from 11 tasks

Task 1.
Transform the motion equation into the state space.

Task 3.
Investigate the stability of the system with forced vibration.

Task 6.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec.

Task 7.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by **Euler´s method** with step $h = 0.1$.

Task 8.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by the **Runge-Kutta ´s method** of 4^{th} order with step $h = 0.1$.

TASK

Applied Mathematics (master degree of study)

↓ lower level

A sports car weighing m = 1000 kg moves along an uneven road in time T at a constant speed. Mathematical model of the uneven road is given by function $u(t) = \sin(t)$. Equivalent values of the stiffness and damping of the automobile suspension are k = 2000 N/m and b = 2000 Ns/m.

Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

$$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$$

$$x(0)=0$$

$$\dot{x}(0)=30$$

↓ higher level

A car is moving along an uneven road in time T at a constant speed. Mathematical model of the uneven road is given by function $u(t)$.

Movement of the non spring-loaded masses and spring-loaded masses of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the motion equations for a quarter model of a car with 2 degrees of freedom (Fig.) in the physical space

$$m_1\ddot{x}_1+k_1(x_1-u)-k_2(x_2-x_1)-b_2(\dot{x}_2-\dot{x}_1)=0$$

$$m_2\ddot{x}_2+k_2(x_2-x_1)+b_2(\dot{x}_2-\dot{x}_1)=0$$

$$x_1(0)=0\ ,\ \dot{x}_1(0)=0$$

$$x_2(0)=0\ ,\ \dot{x}_2(0)=0$$

T A S K

Applied Mathematics (master degree of study)

↓ lower level

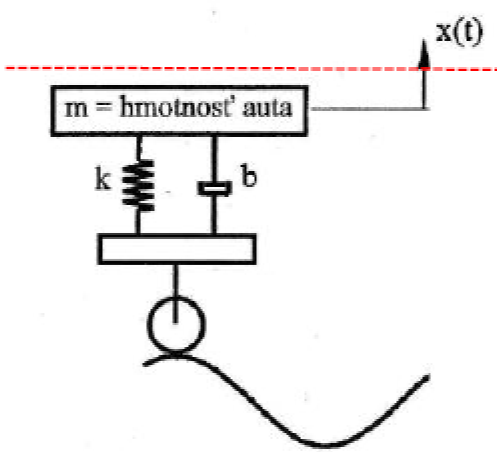


Fig. Quarter model with **1 degree of freedom** of a car moving on uneven road

where

$x = x\,(t)$ – a deflection of a car moving on an uneven road

↓ higher level

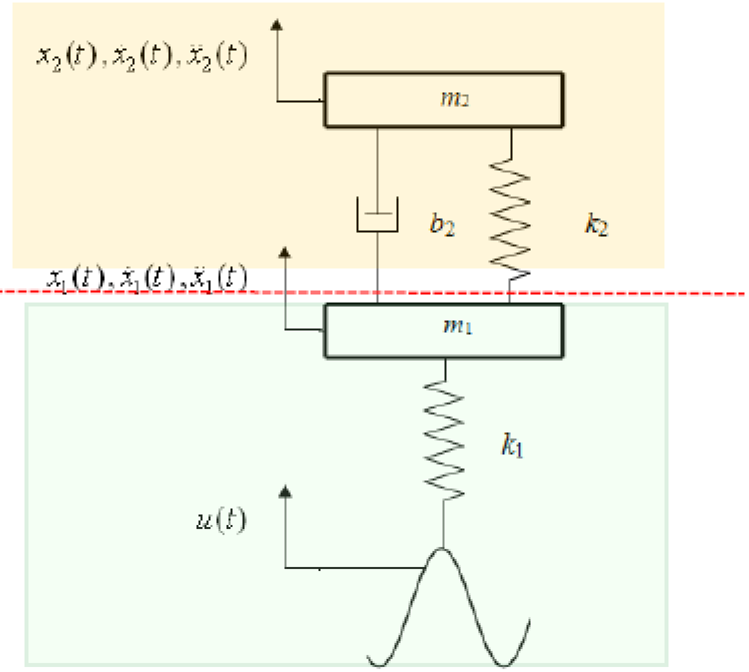


Fig. Quarter model **with 2 degrees of freedom** of a car moving on uneven road

where

- $x_i = x_i\,(t)\,,\ x_i' = x_i'\,(t)\ ,\ x_i'' = x_i''(t)\ ,\ u = u(t)$
- the mathematical model of an uneven road

  $u(t) = 0{,}05\sin(t)$ – the amplitude $A = 0{,}05\ m$
- non spring-loaded masses of a car: $x_1(t)\,, m_1,\ k_1$
- spring-loaded masses of a car: $x_2(t)\,, m_2,\ k_2\,, b_2$

Applied Mathematics (higher level)

The most important tasks from 8 tasks

Task 1.
Transform the motion equation into the state space.

Task 3.
Investigate the stability of the system with forced vibration.

Task 6.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec.

Task 7.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by **Euler´s method** with step $h = 0.1$.

Task 8.
Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec by the **Runge-Kutta ´s method** of 4^{th} order with step $h = 0.1$.

What can our students use to solve PBL?

- software Wolfram Mathematica
- table of formulas of numerical methods
- Wolfram Mathematica type file, containing short programms
  – numerical methods
  – graphs of numerical methods
  – evaluation of errors in numerical solution.

  Each student is "composing" his solution of the project using these programms.

The table of formulas of numerical methods

NUMERICAL METHODS – MULTISTEP

ADAMS – BASHFORTH $(m+1)^{st}$ order		ADAMS – MOULTON $(m+1)^{st}$ order	
Differential equation	**System of differential equations**	**Differential equation**	**System of differential equations**
$t_{i+1}=t_i+h$ $x_{i+1}=x_i+h\cdot k$ $i=0,1,\dots n$ $k=\sum_{j=0}^{m}\beta_{mj}\cdot f_{i-j}$ $f_{i-j}=f(t_{i-j},x_{i-j})$	$t_{i+1}=t_i+h$ $\bar{z}_{i+1}=\bar{z}_i+h\cdot\bar{k}$ $i=0,1,\dots,n$ $\bar{k}=\sum_{j=0}^{m}\beta_{mj}\cdot\bar{f}_{i-j}$ $\bar{f}_{i-j}=A\cdot\bar{z}_{i-j}+\bar{g}(t_{i-j})$	$t_{i+1}=t_i+h$ $\boxed{x_{i+1}}=x_i+h\cdot k$ $i=0,1,\dots n$ $k=\sum_{j=0}^{m}\beta^{*}_{mj}\cdot f_{i-j+1}$ $f_{i-j+1}=f(t_{i-j+1},x_{i-j+1})$	$t_{i+1}=t_i+h$ $\boxed{\bar{z}_{i+1}}=\bar{z}_i+h\cdot\bar{k}$ $i=0,1,\dots,n$ $\bar{k}=\sum_{j=0}^{m}\beta^{*}_{mj}\cdot\bar{f}_{i-j+1}$ $\bar{f}_{i-j+1}=A\cdot\bar{z}_{i-j+1}+\bar{g}(t_{i-j+1})$
$k=\left(\beta_{m0}\cdot f(t_i,x_i)+\beta_{m1}\cdot f(t_{i-1},x_{i-1})+\dots\beta_{mm}\cdot f(t_{i-m},x_{i-m})\right)$		$k=\beta^{*}_{m0}\cdot f\left(t_{i+1},\boxed{x_{i+1}}\right)+\beta^{*}_{m1}\cdot f(t_i,x_i)+\dots\beta^{*}_{mm}\cdot f(t_{i-m+1},x_{i-m+1})$	
$\bar{k}=\left(\beta_{m0}\cdot\bar{f}(t_i,\bar{z}_i)+\beta_{m1}\cdot\bar{f}(t_{i-1},\bar{z}_{i-1})+\dots\beta_{mm}\cdot\bar{f}(t_{i-m},\bar{z}_{i-m})\right)$		$\bar{k}=\beta^{*}_{m0}\cdot\bar{f}\left(t_{i+1},\boxed{\bar{z}_{i+1}}\right)+\beta^{*}_{m1}\cdot\bar{f}(t_i,\bar{z})+\dots\beta^{*}_{mm}\cdot\bar{f}(t_{i-m+1},\bar{z}_{i-m+1})$	

β_{mj}

m	j = 0	j = 1	j = 2	j = 3	error
0	1	–	–	–	O(h)
1	$\frac{3}{2}$	$-\frac{1}{2}$	–	–	$O(h^2)$
2	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	–	$O(h^3)$
3	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$	$O(h^4)$

β^{*}_{mj}

m	j = 0	j = 1	j = 2	j = 3	error
0	1	–	–	–	O(h)
1	$\frac{1}{2}$	$\frac{1}{2}$	–	–	$O(h^2)$
2	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	–	$O(h^3)$
3	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	$O(h^4)$

NUMERICAL METHODS – ONESTEP

Method	Differential equation	System of differential equations	error
	$y_{i+1}=y_i+h\cdot k$ $x_{i+1}=x_i+h$ $i=0,1,\dots,n$	$\bar{z}_{i+1}=\bar{z}_i+h\cdot\bar{k}$ $t_{i+1}=t_i+h$ $i=0,1,\dots,n$	
EULER	$k=f(x_i,y_i)$	$\bar{k}=A\cdot\bar{z}_i+\bar{g}(t_i)$	O(h)
1^{st} modified EULER	$k_1=f(x_i,y_i)$ $k_2=f(x_i+\frac{h}{2},y_i+\frac{h}{2}.k_1)$ $k=k_2$	$\bar{k}_1=A\cdot\bar{z}_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot\left(\bar{z}_i+\frac{h}{2}\cdot\bar{k}_1\right)+\bar{g}\left(t_i+\frac{h}{2}\right)$ $\bar{k}=\bar{k}_2$	$O(h^2)$
2^{nd} modified EULER HEUN´S method.	$k_1=f(x_i,y_i)$ $k_2=f(x_i+h,y_i+h\cdot k_1)$ $k=\frac{k_1+k_2}{2}$	$\bar{k}_1=A\cdot\bar{z}_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot(\bar{z}_i+h\cdot\bar{k}_1)+\bar{g}(t_i+h)$ $\bar{k}=\frac{\bar{k}_1+\bar{k}_2}{2}$	$O(h^2)$
RUNGE – KUTTA 3^{rd} order	$k_1=f(x_i,y_i)$ $k_2=f(x_i+\frac{h}{3},y_i+\frac{h}{3}k_1)$ $k_3=f(x_i+\frac{2}{3}h,y_i+\frac{2}{3}h\cdot k_2)$ $k=\frac{k_1+3k_3}{4}$	$\bar{k}_1=A\cdot\bar{z}_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot\left(\bar{z}_i+\frac{h}{3}\cdot\bar{k}_1\right)+\bar{g}\left(t_i+\frac{h}{3}\right)$ $\bar{k}_3=A\cdot\left(\bar{z}_i+\frac{2}{3}h\cdot\bar{k}_2\right)+\bar{g}\left(t_i+\frac{2}{3}h\right)$ $\bar{k}=\frac{\bar{k}_1+3\bar{k}_3}{4}$	$O(h^3)$
RUNGE – KUTTA 4^{st} order	$k_1=f(x_i,y_i)$ $k_2=f(x_i+\frac{h}{2},y_i+\frac{h}{2}k_1)$ $k_3=f(x_i+\frac{h}{2},y_i+\frac{h}{2}k_2)$ $k_4=f(x_i+h,y_i+h\cdot k_3)$ $k=\frac{k_1+2k_2+2k_3+k_4}{6}$	$\bar{k}_1=A\cdot\bar{z}_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot\left(\bar{z}_i+\frac{h}{2}\cdot\bar{k}_1\right)+\bar{g}\left(t_i+\frac{h}{2}\right)$ $\bar{k}_3=A\cdot\left(\bar{z}_i+\frac{h}{2}\cdot\bar{k}_2\right)+\bar{g}\left(t_i+\frac{h}{2}\right)$ $\bar{k}_4=A\cdot(\bar{z}_i+h\cdot\bar{k}_3)+\bar{g}(t_i+h)$ $\bar{k}=\frac{\bar{k}_1+2\bar{k}_2+2\bar{k}_3+\bar{k}_4}{6}$	$O(h^4)$

Mathematica – the example od short programmes

BASIC SCHEME
ONESTEP AND MULTISTEP NUMERICAL METHODS

```
Clear[t, A, g, z, h, a, b, pin]
A = *;
g[t_] = *;
t[0] = *;
z[0] = { * };
a = *;
b = *;
h = *;
pin = (b - a) / h;
Print["A = ", A // MatrixForm]
Print[]
Print["g[t] = ", g[t] // MatrixForm]
Print[]
Print["a = ", a, "        b = ", b, "       h = ", h]

Eigenvalues[A]

Eigenvectors[A]
```

NUMERICAL SOLUTION FOR ONESTEP METHODS

```
Clear[k1, k2, k3, k4, k]
Do[
    k = *;
    z[i + 1] = z[i] + h * k;
    t[i + 1] = t[i] + h // N,
  {i, 0, pin}]
Print[]
Print["Numericke riesenie"]
TableForm[Table[{i, NumberForm[t[i], 8], NumberForm[z[i][[1]], 8],
    NumberForm[z[i][[2]], 8]}, {i, 0, pin}],
     TableHeadings → {None, {"i", "ti", "z1(t)", "z2(t)"}},
     TableSpacing → {1, 3}]
```

FORMULAS FOR ONESTEP METHODS

```
k = A.z[i] + g[t[i]]
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h/2 * k1) + g[t[i] + h/2];
k = k2;
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h * k1) + g[t[i] + h];
k = (k1 + k2)/2;
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h/3 * k1) + g[t[i] + h/3];
k3 = A.(z[i] + 2 h/3 * k2) + g[t[i] + 2 h/3];
k = (k1 + 3 k3)/4;
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h/2 * k1) + g[t[i] + h/2];
k3 = A.(z[i] + h/2 * k2) + g[t[i] + h/2];
k4 = A.(z[i] + h * k3) + g[t[i] + h];
k = (k1 + 2 k2 + 2 k3 + k4)/6;
```

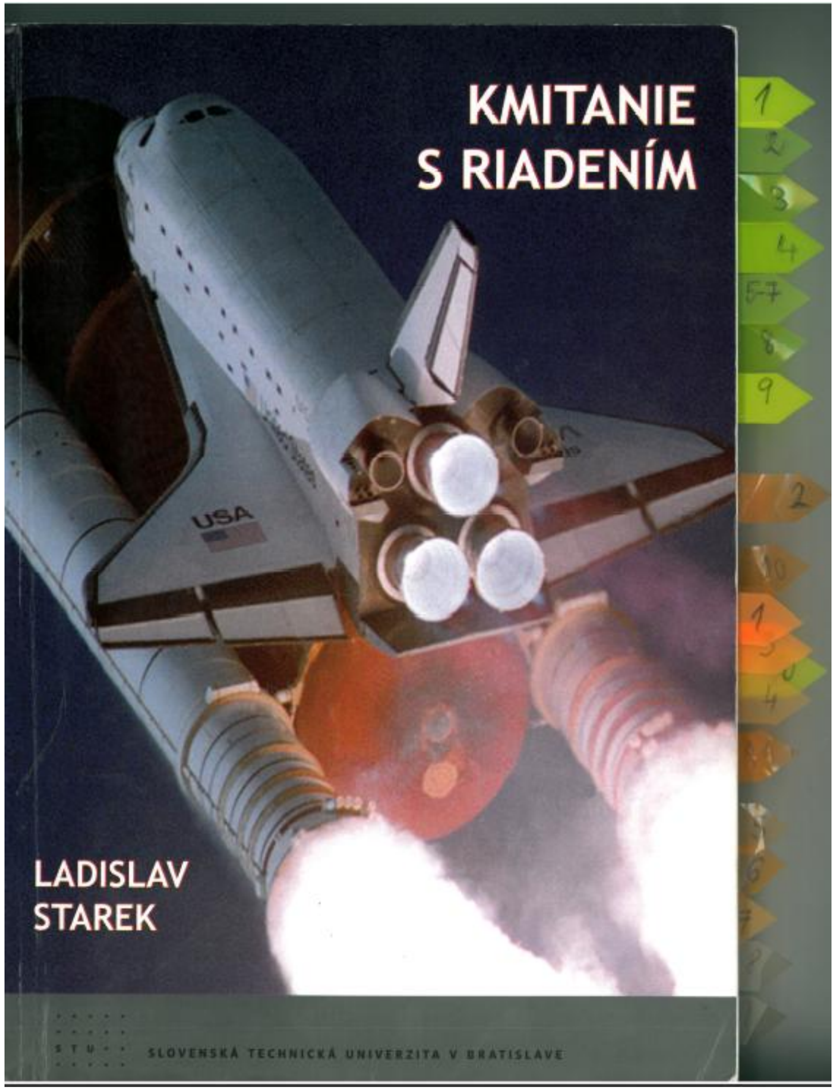
GRAPHICAL SOLUTION FOR ONESTEP METHODS

```
Clear[g1, body1]
body1 = Table[{t[i], z[i][[1]]}, {i, 0, pin}];
Print[" Graf z1(t)"]
g1 = ListPlot[body1, PlotStyle → {PointSize[0.03], Red}]

Clear[g2, body2]
body2 = Table[{t[i], z[i][[2]]}, {i, 0, pin}];
Print[" Graf z2(t)"]
g2 = ListPlot[body2, PlotStyle → {PointSize[0.03], Green}]

Print[" Graf z1(t) , z2(t)"]
Show[g1, g2]
```

Comparison of numerical solutions

```
T = Table[{i, NumberForm[t[i], 8], NumberForm[Abs[z[i][[1]] - zp1[t[i]]], 8],
    NumberForm[Abs[z[i][[2]] - zp2[t[i]]], 8]}, {i, 0, pin}];
TableForm[T, TableHeadings → {None, {"i", "ti", " |zp1(t)-z1(t)|", " |zp2(t)-z2(t)|"}},
    TableSpacing → {1, 3}]
```

Future plans



THANK YOU
FOR YOUR ATTENTION

SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA