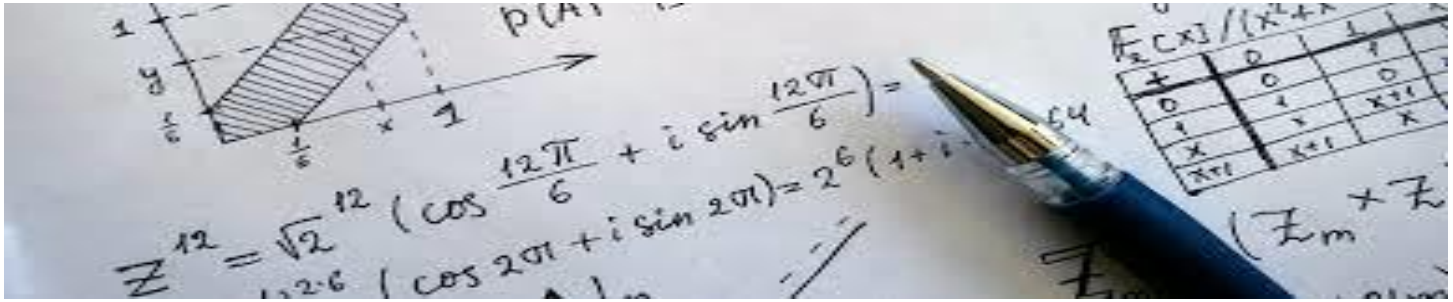

PBL FROM BACHELOR TO ENGINEER

(About how to use one PBL in subjects

Mathematics I at the bachelor degree of study and

Applied Mathematics at the master degree of study)

SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA

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INSTITUTE OF MATHEMATICS AND PHYSICS

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2nd Project Meeting
28 – 29 September, 2023
University of La Laguna

Pythagoras
2021-1-RO01-KA220-HED-000032258

PBL – Why we need it in teaching mathematics?

- ✓ PBLs demonstrate where and how Mathematics can be used continually in special technical subjects
- ✓ students learn to work with

	Mathematics	Technical subject
different variables	$x \longrightarrow y(x)$	$t \longrightarrow u(\mathrm{t})$
constants denoted generally	g = 9,81 m/s^2	g (= 9,81 m/s^2)
different notation of differential equation	$y'' = y''(x)$	$\frac{d^2u}{dt^2} = u''(t) = \ddot{u}(t)$

PBL – Why we need it in teaching mathematics?

- ✓ students become aware of the parallel in the conceptual apparatus of Mathematics and technical subject

Mathematics	Technical subject
the stationary points of a function	the equilibrium position of a system
eigenvalues of the matrix	the stability of a system with forced oscillations
transformation of the motion equation into a system of linear differential equations of the first order	transformation of the motion equation into a state space

- ✓ students better understand the importance of numerical methods in engineering education
- ✓ students have opportunity to test „acquired knowledge“ directly on practical applied problems

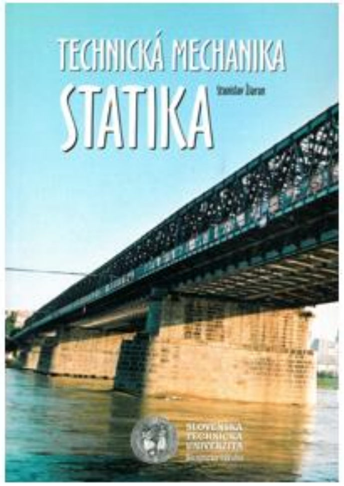
⇓

These problems **must be introduced** into teaching mathematics

since the first year of study at the university

PBL – Where can we find practical problems?

It is necessary to select problems for PBL from the lecture notes for specialised subjects taught at the faculty.



Why?

The advantage of these applications is

that once in the further study

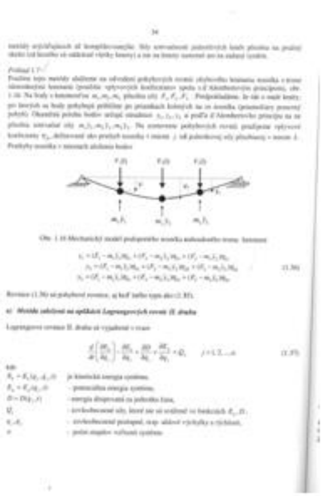
students will “**remember them !** ”,

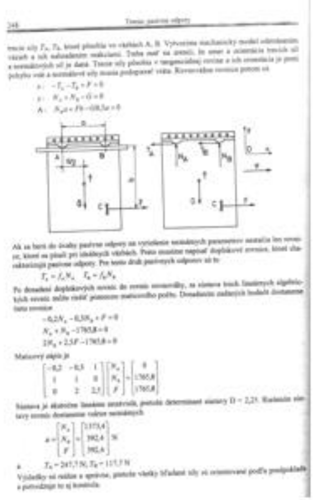
for example **–** special technical text book

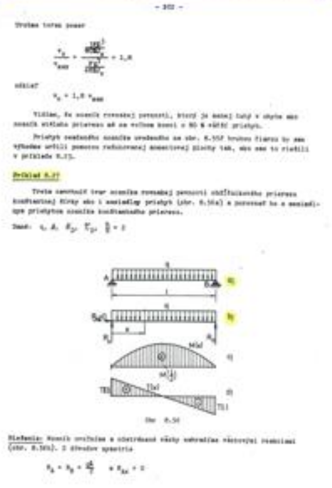
for study programme

Applied Mechanics and Mechatronics

at the FME STU in Bratislava

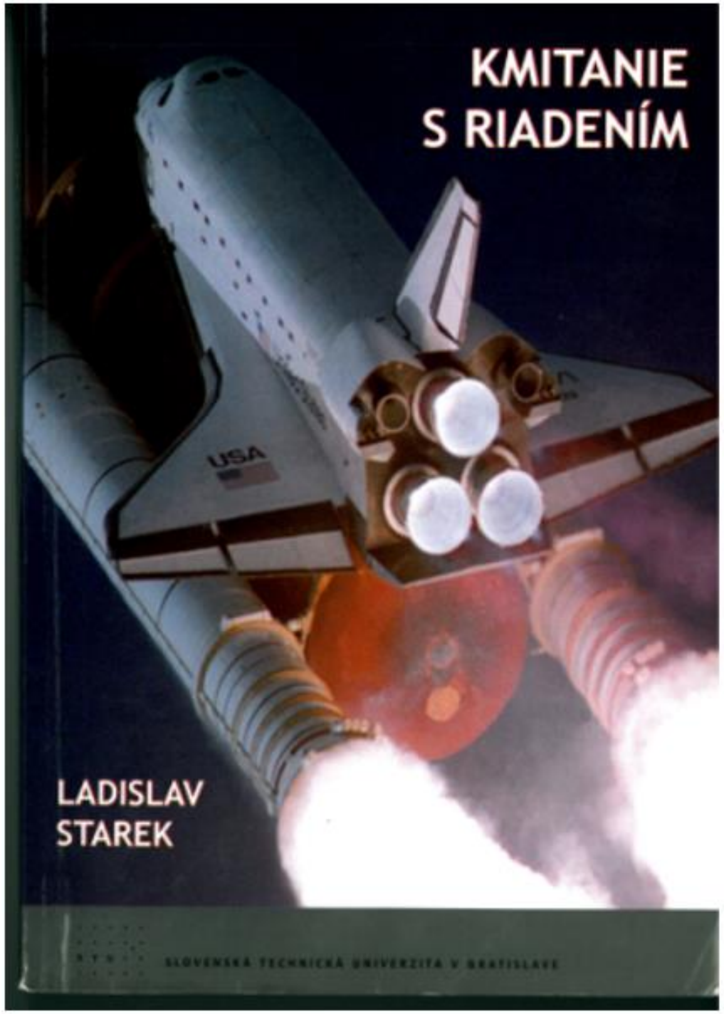






PBL FROM BACHELOR TO ENGINEER

The following PBL was selected from this lecture notes.



61

Ak predpokladáme, že $u = U_u e^{j\omega t}$, bude

$$m\ddot{x} + b\dot{x} + kx = (j\omega b + k)U_u e^{j\omega t} \tag{3.16}$$

Ak porovnáme túto rovnicu s rovnicou (3.3), vidíme, že obidve rovnice budú rovnaké, ak položíme $F_u = U_u(j\omega b + k)$ a $\varphi = 0$. Preto podľa (3.6a) bude

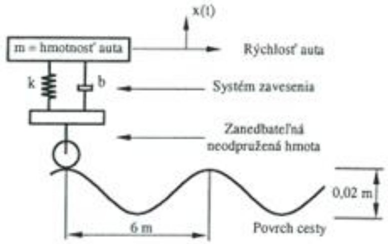
$$\frac{|X|}{U_u} = \sqrt{\frac{1 + (2\eta b_p)^2}{(1-\eta^2)^2 + (2\eta b_p)^2}} \tag{3.17}$$

Vzťah (3.17) vyjadruje pomer maximálnej amplitúdy odozvy k amplitúde vstupu, resp. odpruženej hmoty v závislosti od naladenia.

Príklad 3.2

Bežným príkladom kmitania s kinematickým budením je pohyb automobilu po nerovnej ceste, resp. pohyb lietadla po rozbehovej dráhe. V tomto príklade treba vyšetriť vplyv rýchlosti a hmotnosti športového automobilu na amplitúdu kmitania. Ekvivalentné hodnoty tuhosti a tlmenia odpruženia automobilu sú 4.10^5 N/m, 2.10^4 Ns/m hmotnosť je 1007 kg. Matematický model cesty je daný vzťahom $y(t) = 0{,}01 \sin \omega_b t$. Frekvencia budenia od cesty je $f_b = v/\lambda$, kde v je rýchlosť automobilu v km/h a λ je vlnová dĺžka nerovnosti v km. Platí, že $f = \omega / 2\pi$, a po dosadení máme

$$\omega_b = 2\pi v/\lambda = 2\pi v/(3600.0{,}006) = 0{,}291 \text{ rad/s}$$



Obr. 3.9 Štvrtinový mechanický model automobilu pohybujúci sa konštantnou rýchlosťou po nerovnej ceste (nerovnosti cesty sú pre jednoduchosť aproximované sinusoidou)

Pre rýchlosť automobilu $v = 20$ km/h frekvencia budenia od cesty má hodnotu $\omega_b = 5{,}818$. VUF automobilu je

$$\Omega_0 = \sqrt{4.10^5/1007} = 19{,}93 \text{ rad/s}$$

a teda naladenie

$$\eta = 5{,}818/19{,}93 = 0{,}292$$

Pomerné tlmenie má potom hodnotu

Safe driving of passengers traveling on uneven road

Travellers in cars are exposed to fatigue and unwanted vibrations that affect their health. Driver fatigue affects his performance and reactions, which increases the risk of a traffic accident. Undesirable vibrations, the frequency of which depends on the speed of the car, occur when the car drives on an uneven road (a road with a different profile)

– in the **vertical** direction (causing **floating**)

– in the **longitudinal** direction (causing **rocking**)

– in the **transverse** direction (causing **swinging**).

For **vertical direction** of oscillation, it is necessary to avoid the **frequency band** in the range of **4-8 Hz** (the natural frequency of the human organism in the abdomen), which causes nauseacking.

The safety of the car and the comfort of passengers are improved by the suspension system of the car and properly designed damping.

Several **models of the car - quarter, half** and **full** are used in order to simulate the oscillation of the car, it's non spring-loaded and spring-loaded masses when driving on an uneven road at different speeds.

The behavior of the car when driving on an uneven road allows us to know the solution of the equations of motion for the corresponding model of the car in the physical and state space.

TASK

A sports car weighing m = 1000 kg is moving along an uneven road in time T at a constant speed. Mathematical model of the uneven road is given by function $u(t)$ = sin(t). Equivalent values of the stiffness and damping of the automobile suspension are k = 2000 N/m and b = 2000 Ns/m. Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

Mathematics I (**bachelor degree** of study) ↓	***Applied Mathematics** / lower level* (**master degree** of study) ↓
$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$ $x(0)=0$ $\dot{x}(0)=0$	$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$ $x(0)=0$ $\dot{x}(0)=30$

where

$x = x\,(t)$ – a deflection of a car moving on an uneven road

$x' = x'(t)\ ,\ x'' = x''(t)\ ,\ u = u(t)$

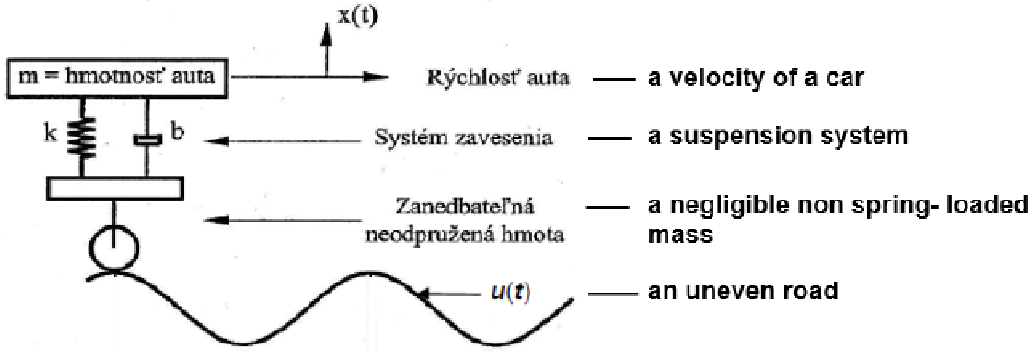


Fig. Quarter mechanical model (with 1 degree of freedom) of a car moving on uneven road

TASK

Mathematics I
(**bachelor degree** of study)

↓

Tasks

- Adjust the motion equation to the basic form.
- **Find the deflection of a car moving on an uneven road in time *T*.**

  – an **analytical** solution
    - the general solution
    - the particular solution determined by Cauchy initial conditions.
- Interpret the results about a car moving on an uneven road in physical units

$$x(3) = 0{,}54$$
$$\dot{x}(3) = -1{,}07$$
$$\ddot{x}(3) = -0{,}64$$

Applied Mathematics / *lower level*
(**master degree** of study)

↓

The **most important** tasks from 11 tasks

- Transform the motion equation into the state space.
- Investigate the stability of the system with forced oscillations.
- **Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec**

  – an **analytical** solution

  – an **approximate** solution with step h = 0.1
    - by the Euler´s method
    - by the Runge-Kutta´s method of the 4th order.
- Compare accuracy of used numerical methods by calculation and graphically.
  (Use global trunction error of a numerical method $|x_i(t) - x_i|$.)

T
A
S
K

Applied Mathematics

(master degree of study)

↓ lower level

A sports car is moving along an uneven road in time *T* at a constant speed.
Mathematical model of the uneven road is given by function $u(t) = \sin(t)$.

Movement of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the equation of motion for a quarter model of a car with 1 degree of freedom (Fig.) in the physical space

$$m\ddot{x} + b(\dot{x} - \dot{u}) + k(x - u) = 0$$

$$x(0) = 0$$

$$\dot{x}(0) = 30$$

with parameters

– the weight of the car m = 1000 kg

– equivalent values of the stiffness and damping of the automobile suspension are k = 2000 N/m and b = 2000 Ns/m.

↓ higher level

A car is moving along an uneven road in time *T* at a constant speed.
Mathematical model of the uneven road is given by function $u(t) = A\sin(t)$ with the amplitude A = 0,05 m.

Movement of non spring-loaded masses and spring-loaded masses of the car on the uneven road represents a real system with forced oscillation.

The behavior of the car (oscillating system) when driving on an uneven road allows us to know the solution of the motion equations for a quarter model of a car with 2 degrees of freedom (Fig.) in the physical space

$$m_1\ddot{x}_1 + k_1(x_1 - u) - k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) = 0$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) = 0$$

$$x_1(0) = 0 \;,\; \dot{x}_1(0) = 0$$

$$x_2(0) = 0 \;,\; \dot{x}_2(0) = 0$$

with parameters

- non spring-loaded masses
  - a displacement $x_1(t)$ /m
  - a mass m_1 = 386,48 / kg
  - a radial stiffness coeffic. of the tyre $k_1 = 1{,}2528.10^6$ / Nm^{-1}
- spring-loaded masses
  - a displacement $x_2(t)$ /m
  - a mass m_2 = 2628,7 / kg
  - a radial stiffness coeffic of the tyre $k_2 = 2{,}0597.10^5$ / Nm^{-1}
  - a damping coeffic. of oil shock absorber $b_2 = 6{,}722.10^3$/ Nsm^{-1}

T A S K

Applied Mathematics (master degree of study)

↓ **lower level**

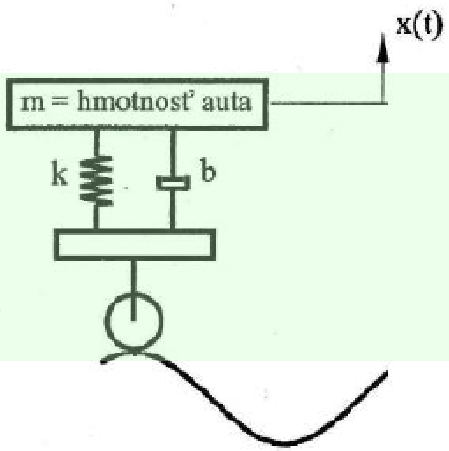


Fig. **Quarter model** with **1 degree of freedom** of a car moving on uneven road

where

$x = x(t)$ – a deflection of a car moving on an uneven road

↓ **higher level**

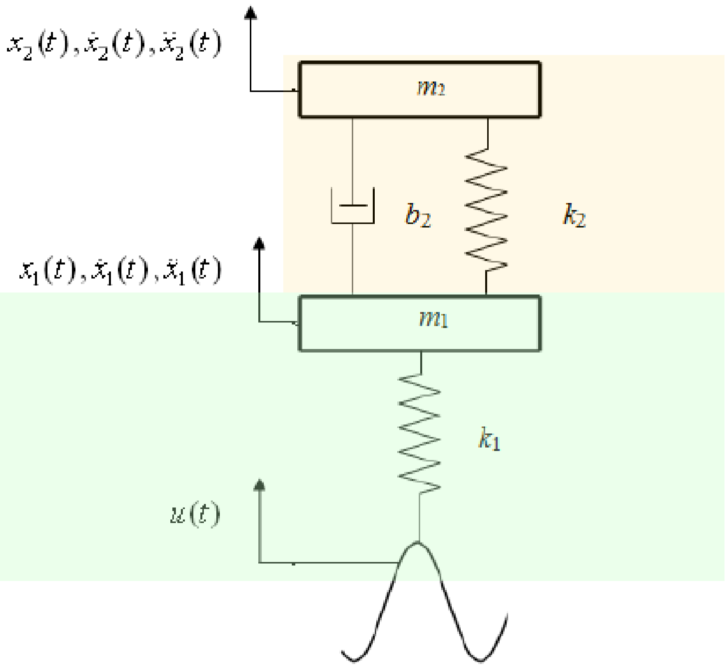


Fig. **Quarter model** with **2 degrees of freedom** of a car moving on uneven road

where

- $x_i = x_i(t)$, $x_i' = x_i'(t)$, $x_i'' = x_i''(t)$
- the mathematical model of an uneven road $u = u(t)$
- non spring-loaded masses of a car: $x_1(t), m_1, k_1$
- spring-loaded masses of a car: $x_2(t), m_2, k_2, b_2$

TASK

Applied Mathematics (master degree of study)

↓ lower level

The **important** tasks from 11 tasks

- Transform the motion equation into the state space.
- Investigate the stability of the system with forced oscillations.
- **Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec**

  – an **analytical** solution

  – an **approximate** solution with step h = 0.1
    - by the Euler´s method
    - by the Runge-Kutta´s method of the 4th order.
- Compare accuracy of used numerical methods by calculation and graphically.
  (Use global trunction error of a numerical method $|x_i(t) - x_i|$.)

↓ higher level

The **important** tasks from 9 tasks.

- Transform the motion equation into the state space.
- Investigate the stability of the system with forced oscillations.
- **Display the deflection, velocity and acceleration of the non spring-loaded and spring-loaded masses of a car at time $t \in [0,10]$ in sec.**

  – there **is not an analytical** solution

  – an **approximate** solution
    - by the NDSolve command.
    - by the Euler´s method with step $h = 0.01$
      (If this method is not stable for he given step h, modify step h so that it was stable.)

  Can we say, based on the graphs, that after a certain time the acceleration of the non spring-loaded and spring-loaded masses of the car are dampened, i.e. *the car's movement on the uneven road will be stabilized?*
- Calculate and compare approximate values of accelerations at the time $t = 1$ sec.
  (Use Non-symmetric formula backward of the order $O(h^2)$, $h = 0.001$

SOLUTION

Mathematics I (bachelor degree of study)

Tasks

- Adjust the motion equation to the basic form.

$$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0 \rightarrow \ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=\frac{k}{m}x+\frac{b}{m}\dot{u}$$

After substituting of input values

$$\ddot{x}+2\dot{x}+2x=2\sin(t)+2\cos(t)$$

- **Find the deflection of a car moving on an uneven road in time *T*.**

An **analytical** solution

- the general solution

$$x(t)=c_1e^{-t}\sin(t)+c_2e^{-t}\cos(t)-\frac{2}{5}\cos(t)+\frac{6}{5}\sin(t),\;\; c_1,c_2\in R\,,\;\; t\in[0,T]\,, T>0$$

- the particular solution determined by Cauchy initial conditions.

$$x_P(t)=-\frac{4}{5}e^{-t}\sin(t)+\frac{2}{5}e^{-t}\cos(t)-\frac{2}{5}\cos(t)\;+\frac{6}{5}\sin(t)\;,\;\; t\in[0,T]\,,T>0$$

- Interpret the results about a car moving on an uneven road in physical units

$$x(3)=\;\;0{,}54$$
$$\dot{x}(3)=-1{,}07$$
$$\ddot{x}(3)=-0{,}64$$

SOLUTION

↓ lower level

The **important** tasks from 11 tasks

- Transform the motion equation into the state space.

$$m\ddot{x}+b(\dot{x}-\dot{u})+k(x-u)=0$$

$$x(0)=0\ ,\ \dot{x}(0)=30$$

In the matrix form

$$\bar{z}\,'(t)=A.\bar{z}(t)+\bar{f}(t)\ ;\ \bar{z}(0)=(0,30)^T$$

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{pmatrix}\cdot\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}0\\ \frac{k}{m}u+\frac{b}{m}\dot{u}\end{pmatrix}$$

$$\begin{pmatrix}z_1(0)\\ z_2(0)\end{pmatrix}=\begin{pmatrix}0\\ 30\end{pmatrix}$$

After substituting of input values

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ -2 & -2\end{pmatrix}\cdot\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}0\\ 2\sin t+2\cos t\end{pmatrix}$$

$$\begin{pmatrix}z_1(0)\\ z_2(0)\end{pmatrix}=\begin{pmatrix}0\\ 30\end{pmatrix}$$

↓ higher level

The **important** tasks from 9 tasks.

- Transform the motion equation into the state space.

$$m_1\ddot{x}_1+k_1(x_1-u)-k_2(x_2-x_1)-b_2(\dot{x}_2-\dot{x}_1)=0$$

$$m_2\ddot{x}_2+k_2(x_2-x_1)+b_2(\dot{x}_2-\dot{x}_1)=0$$

$$x_1(0)=0\ ,\ \dot{x}_1(0)=0$$

$$x_2(0)=0\ ,\ \dot{x}_2(0)=0$$

In the matrix form

$$\bar{z}\,'(t)=A\cdot\bar{z}(t)+\bar{f}(t)\ ;\ \bar{z}(0)=\bar{0}$$

$$\begin{pmatrix}z_1'\\ z_2'\\ z_3'\\ z_4'\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ \frac{-k_1-k_2}{m_1} & \frac{-b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1}\\ 0 & 0 & 0 & 1\\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & \frac{-k_2}{m_2} & \frac{-b_2}{m_2}\end{pmatrix}\cdot\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}+\begin{pmatrix}0\\ \frac{k_1}{m_1}u(t)\\ 0\\ 0\end{pmatrix}$$

$$(z_1(0),z_2(0),z_3(0),z_4(0))^T=(0,0,0,0)^T$$

After substituting of input values

$$\begin{pmatrix}z_1'\\ z_2'\\ z_3'\\ z_4'\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ -3774.5 & -17.3929 & 532.938 & 17.3929\\ 0 & 0 & 0 & 1\\ 78.3543 & 2.55716 & -78.3543 & -2.55716\end{pmatrix}\cdot\begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}+\begin{pmatrix}0\\ 162.78\sin(t)\\ 0\\ 0\end{pmatrix}$$

$$(z_1(0),z_2(0),z_3(0),z_4(0))^T=(0,0,0,0)^T$$

S O L U T I O N

Applied Mathematics (master degree of study)

↓ **lower level**

The **important** tasks from 11 tasks

- Investigate the stability of the system with forced oscillations.

  This system is stable when all eigenvalues λ of matrix A of the system $\bar{z}\,'(t) = A\,\bar{z}(t)$ are $\lambda = a + ib \in C \wedge a < 0$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

$\lambda \to -1 - i$

$\lambda \to -1 + i$

↓ **higher level**

The **important** tasks from 9 tasks.

- Investigate the stability of the system with forced oscillations.

  This system is stable when all eigenvalues λ of matrix A of the system $\bar{z}\,'(t) = A\,\bar{z}(t)$ are $\lambda = a + ib \in C \wedge a < 0$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3774.5 & -17.3929 & 532.938 & 17.3929 \\ 0 & 0 & 0 & 1 \\ 78.3543 & 2.55716 & -78.3543 & -2.55716 \end{pmatrix}$$

$\lambda \to -9.03318 - 60.5765\, i$

$\lambda \to -9.03318 + 60.5765\, i$

$\lambda \to -0.941841 - 8.17457\, i$

$\lambda \to -0.941841 + 8.17457\, i$

SOLUTION

Applied Mathematics (master degree of study)

↓ lower level

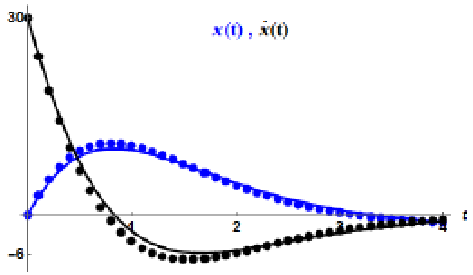
The **important** tasks from 11 tasks

- **Find and display the deflection and velocity of a car moving on an uneven road in time $t \in [0, 4]$ in sec.**

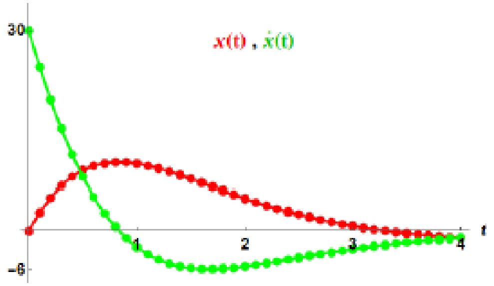
  – an **analytical** solution
  – an **approximate** solution with step h = 0.1
    - by the Euler´s method
    - by the Runge-Kutta´s method of the 4th order.

Graphical comparison

Euler´s method vs. **analyt.** solution



Runge-Kutta´s method vs. **analyt.** solution



↓ higher level

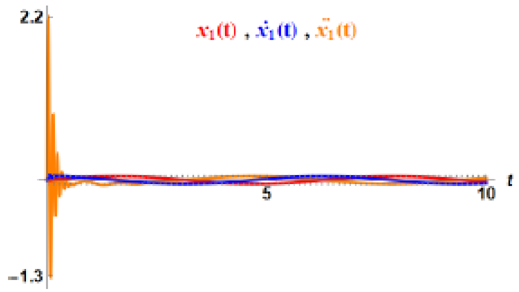
The **important** tasks from 9 tasks.

- **Display the deflection, velocity and acceleration of the non spring-loaded and spring-loaded masses of a car at time t ∈ [0,10] in sec.**

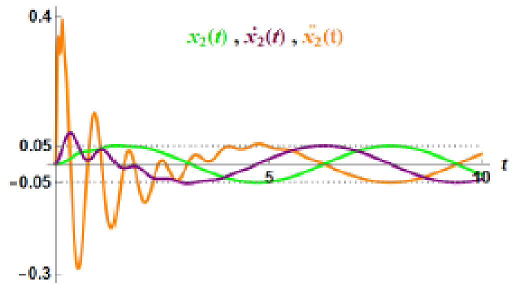
  – there **is not an analytical** solution
  – an **approximate** solution
    - by the NDSolve command.

Can we say that the car's movement on the uneven road will be stabilized?

Accelerations of the **non spring-loaded masses**



and **spring-loaded masses** of a car



are dampened after a certain time, it means **a motion of a car will be stabilized** on an uneven road.

Applied Mathematics / higher level

SOLUTION

The **important** tasks from 9 tasks.

- **Display the deflection, velocity and acceleration of the non spring-loaded and spring-loaded masses of a car at time t ∈ [0,10] in sec.**

  – there **is not an analytical** solution

  – an **approximate** solution by the Euler´s method with step $h = 0.01$

  (If this method is not stable for he given step h, modify step h so that it was stable.)

Non spring-loaded masses of a car

the step $\boldsymbol{h}$ **= 0,01** — the **unstable** method

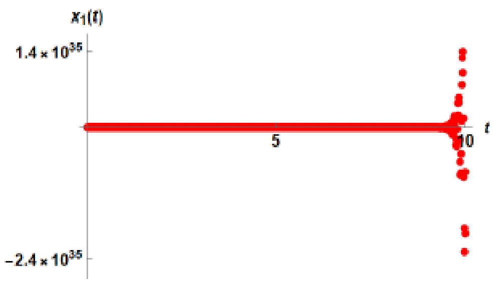
the step $\boldsymbol{h}$ **= 0,001** — the **stable** method

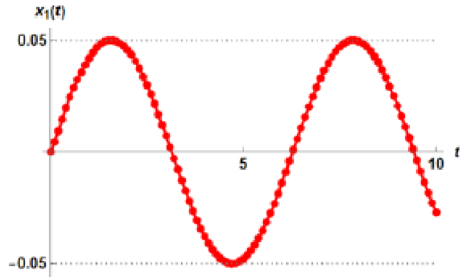
Spring-loaded masses of a car

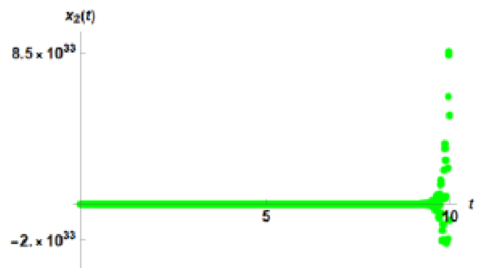
the step $\boldsymbol{h}$ **= 0,01** — the **unstable** method

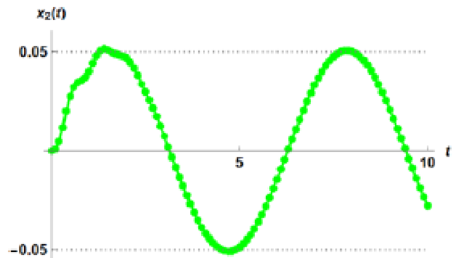
the step $\boldsymbol{h}$ **= 0,001** — the **stable** method

the deflection

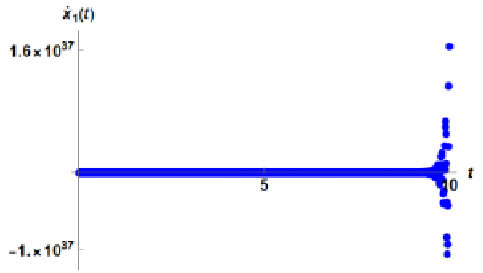


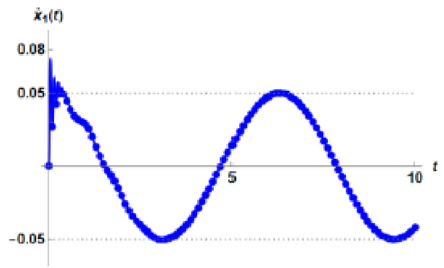


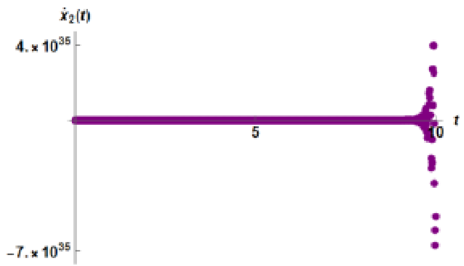


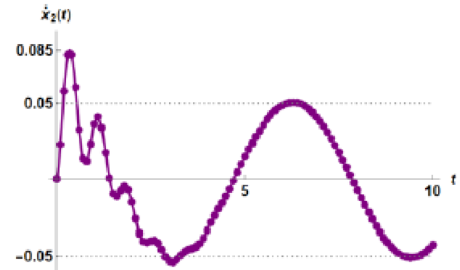


the velocity









S O L U T I O N

Applied Mathematics (master degree of study)

↓ **lower level**

The **most important** tasks from 11 tasks

- Compare accuracy of used numerical methods by calculation and graphically.
  (Use global trunction error of a numerical method $|x_i(t) - x_i|$.)

Comparison by calculation

(We only list every 5th solution)

i	t	$\lvert x(t)-x_E\rvert$	$\lvert \dot{x}(t)-\dot{x}_E\rvert$	$\lvert x(t)-x_{RK4}\rvert$	$\lvert \dot{x}(t)-\dot{x}_{RK4}\rvert$
0	0	0.	0.	0.	0.
5	0.5	0.84869224	1.2518389	0.000017103898	0.000059030232
10	1.	0.65381293	1.576066	4.9829304×10^{-6}	0.000049065915
15	1.5	0.10688962	1.109394	0.000026081243	0.000014363026
20	2.	0.31611341	0.40919656	0.00003286521	0.000014643786
25	2.5	0.46898199	0.12387642	0.000027730321	0.000027660012
30	3.	0.41350121	0.36476462	0.000017824845	0.000027296591
35	3.5	0.27263535	0.37376304	8.8493427×10^{-6}	0.000020429769
40	4.	0.14149862	0.27177548	3.2458854×10^{-6}	0.000012823749

↓ **higher level**

The **important** tasks from 9 tasks.

- Calculate and compare approximate values of accelerations at the time t = 1 sec.
  (Use Non-symmetric formula backward of the order $O(h^2)$, h = 0.001.

Comparison

$$f''(x) \approx \frac{2f(x) - 5f(x-h) + 4f(x-2h) - f(x-3h)}{h^2}$$

$f''(x)$	*masses of a car*	
	non spring-loaded	*spring-loaded*
NDSolve	– 0,022	0,102
Euler´s method	– 0,025	0,107
the difference	0,003	0,005

What can our students use to solve PBL?

- ✓ software Wolfram Mathematica
- ✓ table of formulas of numerical methods
- ✓ Wolfram Mathematica type file containing short programmes
  – numerical methods
  – graphs of numerical methods
  – evaluation of errors in numerical solution.

Each student is “composing” his solution of the project using these programmes.

The table of formulas of numerical methods

NUMERICAL METHODS – MULTISTEP

ADAMS – BASHFORTH (m+1)st order – Differential equation	ADAMS – BASHFORTH (m+1)st order – System of differential equations	ADAMS – MOULTON (m+1)st order – Differential equation	ADAMS – MOULTON (m+1)st order – System of differential equations
$t_{i+1}=t_i+h$ $x_{i+1}=x_i+h\cdot k$ $i=0,1,\dots n$ $k=\sum_{j=0}^{m}\beta_{mj}\cdot f_{i-j}$ $f_{i-j}=f(t_{i-j},x_{i-j})$	$t_{i+1}=t_i+h$ $\bar{z}_{i+1}=\bar{z}_i+h\cdot\bar{k}$ $i=0,1,\dots,n$ $\bar{k}=\sum_{j=0}^{m}\beta_{mj}\cdot\bar{f}_{i-j}$ $\bar{f}_{i-j}=A\cdot\bar{z}_{i-j}+\bar{g}(t_{i-j})$	$t_{i+1}=t_i+h$ $\boxed{x_{i+1}}=x_i+h\cdot k$ $i=0,1,\dots n$ $k=\sum_{j=0}^{m}\beta^{*}_{mj}\cdot f_{i-j+1}$ $f_{i-j+1}=f(t_{i-j+1},x_{i-j+1})$	$t_{i+1}=t_i+h$ $\boxed{\bar{z}_{i+1}}=\bar{z}_i+h\cdot\bar{k}$ $i=0,1,\dots,n$ $\bar{k}=\sum_{j=0}^{m}\beta^{*}_{mj}\cdot\bar{f}_{i-j+1}$ $\bar{f}_{i-j+1}=A\cdot\bar{z}_{i-j+1}+\bar{g}(t_{i-j+1})$
$k=\left(\beta_{m0}\cdot f(t_i,x_i)+\beta_{m1}\cdot f(t_{i-1},x_{i-1})+\dots\beta_{mm}\cdot f(t_{i-m},x_{i-m})\right)$		$k=\beta^{*}_{m0}\cdot f\left(t_{i+1},\boxed{x_{i+1}}\right)+\beta^{*}_{m1}\cdot f(t_i,x_i)+\dots\beta^{*}_{mm}\cdot f(t_{i-m+1},x_{i-m+1})$	
$\bar{k}=\left(\beta_{m0}\cdot\bar{f}(t_i,\bar{z}_i)+\beta_{m1}\cdot\bar{f}(t_{i-1},\bar{z}_{i-1})+\dots\beta_{mm}\cdot\bar{f}(t_{i-m},\bar{z}_{i-m})\right)$		$\bar{k}=\beta^{*}_{m0}\cdot\bar{f}\left(t_{i+1},\boxed{\bar{z}_{i+1}}\right)+\beta^{*}_{m1}\cdot\bar{f}(t_i,\bar{z})+\dots\beta^{*}_{mm}\cdot\bar{f}(t_{i-m+1},\bar{z}_{i-m+1})$	

β_{mj}

m	j = 0	j = 1	j = 2	j = 3	error
0	1	–	–	–	O(h)
1	$\frac{3}{2}$	$-\frac{1}{2}$	–	–	O(h²)
2	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	–	O(h³)
3	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$	O(h⁴)

β^{*}_{mj}

m	j = 0	j = 1	j = 2	j = 3	error
0	1	–	–	–	O(h)
1	$\frac{1}{2}$	$\frac{1}{2}$	–	–	O(h²)
2	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	–	O(h³)
3	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	O(h⁴)

NUMERICAL METHODS – ONESTEP

Method	Differential equation	System of differential equations	error
	$y_{i+1}=y_i+h\cdot k$ $x_{i+1}=x_i+h$ $i=0,1,\dots,n$	$z_{i+1}=z_i+h\cdot\bar{k}$ $t_{i+1}=t_i+h$ $i=0,1,\dots,n$	
EULER	$k=f(x_i,y_i)$	$\bar{k}=A\cdot z_i+\bar{g}(t_i)$	O(h)
1st modified EULER	$k_1=f(x_i,y_i)$ $k_2=f(x_i+\frac{h}{2},y_i+\frac{h}{2}.k_1)$ $k=k_2$	$\bar{k}_1=A\cdot z_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot\left(z_i+\frac{h}{2}\cdot\bar{k}_1\right)+\bar{g}\left(t_i+\frac{h}{2}\right)$ $\bar{k}=\bar{k}_2$	O(h²)
2nd modified EULER HEUN´S method.	$k_1=f(x_i,y_i)$ $k_2=f(x_i+h,y_i+h\cdot k_1)$ $k=\frac{k_1+k_2}{2}$	$\bar{k}_1=A\cdot z_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot(\bar{z}_i+h\cdot\bar{k}_1)+\bar{g}(t_i+h)$ $\bar{k}=\frac{\bar{k}_1+\bar{k}_2}{2}$	O(h²)
RUNGE – KUTTA 3rd order	$k_1=f(x_i,y_i)$ $k_2=f(x_i+\frac{h}{3},y_i+\frac{h}{3}k_1)$ $k_3=f(x_i+\frac{2}{3}h,y_i+\frac{2}{3}h\cdot k_2)$ $k=\frac{k_1+3k_3}{4}$	$\bar{k}_1=A\cdot z_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot\left(z_i+\frac{h}{3}\cdot\bar{k}_1\right)+\bar{g}\left(t_i+\frac{h}{3}\right)$ $\bar{k}_3=A\cdot\left(z_i+\frac{2}{3}h\cdot\bar{k}_2\right)+\bar{g}\left(t_i+\frac{2}{3}h\right)$ $\bar{k}=\frac{\bar{k}_1+3\bar{k}_3}{4}$	O(h³)
RUNGE – KUTTA 4st order	$k_1=f(x_i,y_i)$ $k_2=f(x_i+\frac{h}{2},y_i+\frac{h}{2}k_1)$ $k_3=f(x_i+\frac{h}{2},y_i+\frac{h}{2}k_2)$ $k_4=f(x_i+h,y_i+h\cdot k_3)$ $k=\frac{k_1+2k_2+2k_3+k_4}{6}$	$\bar{k}_1=A\cdot\bar{z}_i+\bar{g}(t_i)$ $\bar{k}_2=A\cdot\left(z_i+\frac{h}{2}\cdot\bar{k}_1\right)+\bar{g}\left(t_i+\frac{h}{2}\right)$ $\bar{k}_3=A\cdot\left(z_i+\frac{h}{2}\cdot\bar{k}_2\right)+\bar{g}\left(t_i+\frac{h}{2}\right)$ $\bar{k}_4=A\cdot(\bar{z}_i+h\cdot\bar{k}_3)+\bar{g}(t_i+h)$ $\bar{k}=\frac{\bar{k}_1+2\bar{k}_2+2\bar{k}_3+\bar{k}_4}{6}$	O(h⁴)

Mathematica – the example of short programmes

BASIC SCHEME

ONESTEP AND MULTISTEP NUMERICAL METHODS

```
Clear[t, A, g, z, h, a, b, pin]
A = *;
g[t_] = *;
t[0] = *;
z[0] = { * };
a = *;
b = *;
h = *;
pin = (b - a) / h;
Print["A = ", A // MatrixForm]
Print[]
Print["g[t] = ", g[t] // MatrixForm]
Print[]
Print["a = ", a, "        b = ", b, "       h = ", h]

Eigenvalues[A]

Eigenvectors[A]
```

NUMERICAL SOLUTION FOR ONESTEP METHODS

```
Clear[k1, k2, k3, k4, k]
Do[
    k = *;
    z[i + 1] = z[i] + h * k;
    t[i + 1] = t[i] + h // N,
  {i, 0, pin}]
Print[]
Print["Numericke riesenie"]
TableForm[Table[{i, NumberForm[t[i], 8], NumberForm[z[i][[1]], 8],
    NumberForm[z[i][[2]], 8]}, {i, 0, pin}],
      TableHeadings → {None, {"i", "ti", "z1(t)", "z2(t)"}},
      TableSpacing → {1, 3}]
```

FORMULAS FOR ONESTEP METHODS

```
k = A.z[i] + g[t[i]]
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h/2 * k1) + g[t[i] + h/2];
k = k2;
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h * k1) + g[t[i] + h];
k = (k1 + k2)/2;
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h / 3 * k1) + g[t[i] + h / 3];
k3 = A.(z[i] + 2 h / 3 * k2) + g[t[i] + 2 h / 3];
k = (k1 + 3 k3)/4;
```

```
k1 = A.z[i] + g[t[i]];
k2 = A.(z[i] + h / 2 * k1) + g[t[i] + h / 2];
k3 = A.(z[i] + h / 2 * k2) + g[t[i] + h / 2];
k4 = A.(z[i] + h * k3) + g[t[i] + h];
k = (k1 + 2 k2 + 2 k3 + k4)/6;
```

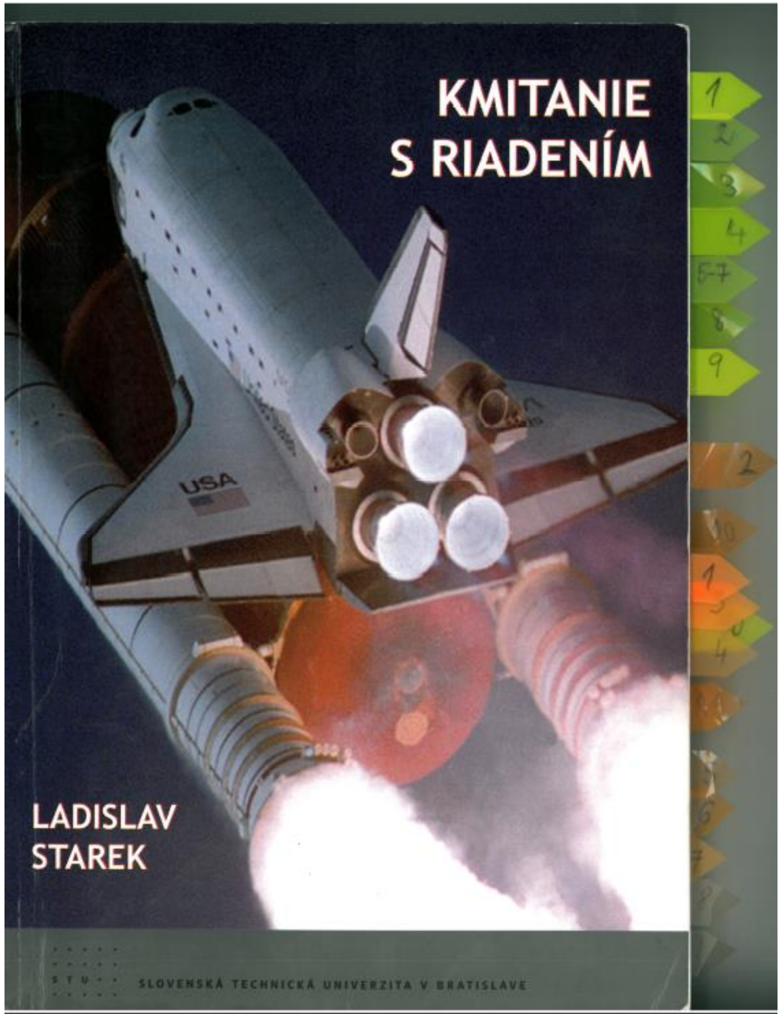
GRAPHICAL SOLUTION FOR ONESTEP METHODS

```
Clear[g1, body1]
body1 = Table[{t[i], z[i][[1]]}, {i, 0, pin}];
Print[" Graf z1(t)"]
g1 = ListPlot[body1, PlotStyle → {PointSize[0.03], Red}]

Clear[g2, body2]
body2 = Table[{t[i], z[i][[2]]}, {i, 0, pin}];
Print[" Graf z2(t)"]
g2 = ListPlot[body2, PlotStyle → {PointSize[0.03], Green}]

Print[" Graf z1(t) , z2(t)"]
Show[g1, g2]
```

Comparison of numerical solutions

```
T = Table[{i, NumberForm[t[i], 8], NumberForm[Abs[z[i][[1]] - zp1[t[i]]], 8],
     NumberForm[Abs[z[i][[2]] - zp2[t[i]]], 8]}, {i, 0, pin}];
TableForm[T, TableHeadings → {None, {"i", "ti", " |zp1(t)-z1(t)|", " |zp2(t)-z2(t)|"}},
      TableSpacing → {1, 3}]
```

How do we want to proceed with PBL next?



We have already selected other marked topics of suitable application problems

- ✓ to show students of the first year at the bachelor study programmes how and where to use Mathematics in special technical subjects
- ✓ to prepare simultaneously Master‘s students of study programme Automobiles and Mobile Working Machine for their diploma thesis

STU

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